NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3206

TORSIONAL VIBRATIONS OF HOLLOW THIN-WALLED CYLINDRICAL BEAMS

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Washington
August 1954

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SUMMARY

The differential equations and boundary conditions appropriate to the analysis of torsional vibrations of hollow thin-walled cylindrical beams are presented. General solutions for the modes and frequencies of cantilever and free-free cylindrical beams of arbitrary doubly symmetrical cross section with uniform wall thickness are derived. Numerical calculations for beams of rectangular cross section are used to determine the influence on torsional vibrations of bending stresses due to torsion and of longitudinal inertia. The inclusion of longitudinal inertia exposes a strong coupling of the torsional vibration with longitudinal vibrations at low plan-form aspect ratios, and some of the frequencies obtained are seen to be actually associated with modes of primarily longitudinal vibration. The solution for cylinders of rectangular cross section is also used to investigate the accuracy of a solution obtained from an analysis of an equivalent four-flange box beam.

INTRODUCTION

Investigations of the transverse vibration of hollow thin-walled cylindrical beams (see, for example, refs. 1 and 2) show that secondary effects such as transverse shear, shear lag, and longitudinal inertia can have appreciable influence on the transverse modes and frequencies of structures. Torsional oscillations are influenced by similar secondary effects, namely, longitudinal stresses which resist the tendency of the cross sections to warp out of their initial planes and the longitudinal inertia which is developed by the warping motion. (The former effect is commonly referred to as "bending stresses due to torsion" and will be so designated throughout this report.)

In order to assess the importance of these secondary effects, a procedure similar to that used in reference 1 has been used in the present paper to analyze the torsional vibrations of thin-walled cylindrical beams. The governing variational principle, differential equations, and boundary conditions are discussed, and general solutions for cantilever

and free-free cylinders of arbitrary doubly symmetrical cross section with uniform wall thickness are presented. Results of numerical calculations based on the solution for the symmetrically vibrating free-free cylinder of rectangular cross section are used to illustrate the influence of bending stresses due to torsion and longitudinal inertia on torsional oscillations. Finally, the results obtained from the calculations for cylinders of rectangular cross section are compared with those obtained from an analysis of an equivalent four-flange box beam.

SYMBOLS

A_F	cross-sectional area of flange
A_o	cross-sectional area enclosed by median line of wall thickness
B_i	parameter defined by equation (29)
C	constant
E	Young's modulus of elasticity
G	shear modulus of elasticity
I_p	mass polar moment of inertia per unit length
J	torsional stiffness constant, $\frac{4A_o^2 t}{p}$
K	parameter, $\frac{p}{L}\sqrt{\frac{E}{G}}$
K_n	Fourier coefficient
L	length of cantilever beam; half-length of free-free beam
N_i	parameter defined by equation (36)
T	maximum kinetic energy
U	maximum strain energy
a	half-depth of rectangular beam
b	half-width of rectangular beam

a_{mn}, b_m	Fourier series coefficients
i,j,m,n	integers
k_{LI}	coefficient of longitudinal inertia, $\frac{p}{L}\sqrt{\frac{J\mu}{I_p}}$
k_T	frequency coefficient, $\omega\sqrt{\frac{I_p L^2}{GJ}}$
p	perimeter of cross section
s	distance along perimeter of cross section (see fig. 1)
t	thickness of effective shear-carrying area
t'	thickness of effective normal-stress-carrying area
t_a	actual wall thickness
$u(x,s)$	displacement in x-direction
$u_F(x)$	displacement of flange in x-direction
x	longitudinal coordinate
γ_{xs}	shear strain
ϵ_x	longitudinal strain
$\theta(x)$	rotation of cross section
λ	Lagrangian multiplier
μ	mass density of beam
ξ	nondimensional longitudinal coordinate, $\frac{x}{L}$
ρ	distance from centroid of cross section to tangent to the median line of wall thickness (see fig. 1)
σ	longitudinal direct stress
τ	shear stress
ω	natural circular frequency of beam

BASIC EQUATIONS

Assumptions

The problem considered herein is that of the natural torsional vibration of a thin-walled hollow cylindrical beam. The cross section of the beam is assumed to be doubly symmetrical so that no transverse vibrations are introduced. The wall thickness of the cylindrical beam is assumed to be made up of an effective normal-stress-carrying thickness t', an effective shear-carrying thickness t, and a thickness t_a associated with the mass properties of the beam. (These three distinct thicknesses are introduced for generality; they allow, for example, the inclusion of stringers which carry only normal stress and not shear.)

For the present analysis, the following simplifying assumptions (also used in ref. 1) are made:

(a) Changes in the size and shape of the cross section are negligible.

(b) Stress and strain are assumed to be uniform across the wall thickness.

(c) The effect of circumferential stress upon longitudinal strain is negligible.

Because cross sections of hollow thin-walled cylinders are more easily sheared out of shape under torsional loads than under bending loads, assumption (a) has more significance in the present torsion problem than it did in the bending problem in reference 1. For torsional vibrations, this assumption restricts the applicability of the analysis to structures that contain closely spaced bulkheads which are rigid in their own planes but offer negligible resistance to warping out of their planes.

In agreement with the assumptions, the distortions of the vibrating beam will be defined by a rotation $\theta(x)$ of the cross section and a longitudinal displacement $u(x,s)$ of each point of the median line of the beam wall. In agreement with the assumption of double symmetry of the cross section, the displacement u must be antiperiodic in the coordinate s with a period equal to one-quarter the perimeter p.

Stress-Strain Relations

The longitudinal and shear strains in terms of the displacements $\theta(x)$ and $u(x,s)$ are

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{1}$$

and

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \rho \frac{d\theta}{dx} \tag{2}$$

where ρ is the distance from the centroid of the cross section to the tangent to the median line (see fig. 1). The corresponding components of stress are

$$\sigma_x = E \frac{\partial u}{\partial x} \tag{3}$$

and

$$\tau_{xs} = G\left(\frac{\partial u}{\partial s} + \rho \frac{d\theta}{dx}\right) \tag{4}$$

In elementary torsion theory, the cross section of the beam is considered to be unrestrained against warping (that is, E = 0) with the result that the longitudinal stress σ_x is zero.

Variational Principle and Energy Relations

In order to use the variational equation

$$\delta(U - T) = 0 \tag{5}$$

in the present analysis, the maximum strain energy U and the maximum kinetic energy T are expressed in terms of the displacements $\theta(x)$ and $u(x,s)$. The variation is taken independently with respect to $\theta(x)$ and $u(x,s)$ with the provision that these displacements satisfy the appropriate geometrical boundary conditions and that $u(x,s)$ be antiperiodic in s with a period $p/4$.

For a beam vibrating in a natural torsion mode, the maximum strain energy is

$$U = \frac{1}{2}\int_0^L \oint Gt\left(\frac{\partial u}{\partial s} + \rho \frac{d\theta}{dx}\right)^2 ds\ dx + \frac{1}{2}\int_0^L \oint Et'\left(\frac{\partial u}{\partial x}\right)^2 ds\ dx \tag{6}$$

The first term in equation (6) represents the strain energy usually considered in elementary theory and the second term represents the strain energy contributed by the restraint of warping.

The maximum kinetic energy is

$$T = \frac{1}{2}\int_0^L \omega^2 I_p \theta^2 dx + \frac{1}{2}\int_0^L \oint \omega^2 \mu t_a u^2 ds\ dx \tag{7}$$

where ω is the natural circular frequency of the particular mode, I_p is the mass polar moment of inertia per unit length of the cross section, and μ is the mass density of the beam. The first term in equation (7) represents the kinetic energy usually considered in elementary theory; the second term represents the kinetic energy of warping or longitudinal motion.

Differential Equations and Boundary Conditions

The substitution of equations (6) and (7) into equation (5) and the application of the calculus of variations results in the simultaneous integrodifferential equations for the displacements $u(x,s)$ and $\theta(x)$,

$$Et' \frac{\partial^2 u}{\partial x^2} + G \frac{\partial t}{\partial s}\frac{\partial u}{\partial s} + Gt \frac{\partial^2 u}{\partial s^2} + G \frac{\partial(\rho t)}{\partial s}\frac{d\theta}{dx} + \mu\omega^2 t_a u = 0 \tag{8}$$

$$\oint Gt\rho\left(\frac{\partial^2 u}{\partial x\ \partial s} + \rho \frac{d^2\theta}{dx^2}\right)ds + I_p\omega^2\theta = 0 \tag{9}$$

together with the boundary conditions at each end of the beam,

$$\oint Et' \frac{\partial u}{\partial x}\delta u\ ds = 0 \tag{10}$$

and

$$\oint Gt\left(\frac{\partial u}{\partial s} + \rho \frac{d\theta}{dx}\right)\rho\ ds\ \delta\theta = 0 \tag{11}$$

At a fixed end, both boundary equations (10) and (11) are satisfied since the geometrical boundary conditions require that both δu and $\delta\theta$ be zero. At a free end, since both δu and $\delta\theta$ are arbitrary, equations (10) and (11) yield the natural boundary conditions

$$Et' \frac{\partial u}{\partial x} = 0 \tag{12}$$

and

$$\oint Gt\left(\frac{\partial u}{\partial s} + \rho \frac{d\theta}{dx}\right)\rho \, ds = 0 \tag{13}$$

Equation (12) expresses the condition of zero longitudinal direct stress and equation (13) expresses the condition of zero net torque on the free end.

Although the integrodifferential equations (8) and (9) were obtained by use of the variational principle, these equations can, of course, be obtained directly from consideration of equilibrium conditions.

GENERAL SOLUTIONS FOR CYLINDERS OF UNIFORM WALL THICKNESS

In this section, the exact solutions are presented for cylinders of uniform wall thickness; that is, t', t, and t_a are each uniform around the cross section. These solutions are obtained by means of Fourier series and the variational principle (eq. (5)) - a more expedient approach than a direct attack on the integrodifferential equations (8) and (9) and their associated boundary conditions.

Cantilever Beam

For a cantilever beam (fig. 1(c)), the geometrical conditions are $u(0,s) = \theta(0) = 0$. The natural boundary conditions, equations (12) and (13), are satisfied as a result of the variational process. The displacements $\theta(x)$ and $u(x,s)$ are assumed to be given by the expressions

$$\theta(x) = C + \sum_{m=1,3,5}^{\infty} b_m \cos \frac{m\pi x}{2L} \tag{14}$$

and

$$u(x,s) = \sum_{m=1,3,5}^{\infty} \sum_{n=2,4,6}^{\infty} a_{mn} \sin \frac{m\pi x}{2L} \sin \frac{2n\pi s}{p} \tag{15}$$

The choice of the particular trigonometric functions used in the expressions for u and θ was guided by consideration of the orthogonality required for the simplification of the strain-energy expressions. The condition $u(0,s) = 0$ is satisfied by each term of equation (15). The

constant C in the expression for $\theta(x)$ is required to allow $\theta(L)$ to be unrestricted. The condition

$$\theta(0) = C + \sum_{m=1,3,5}^{\infty} b_m = 0 \tag{16}$$

is introduced into the variational procedure by means of the Lagrangian multiplier method.

Substitution of equations (14) and (15) into equations (6) and (7) yields

$$U - T = \frac{Et_1'}{2}\int_0^L \oint \left(\sum_{m=1,3,5}^{\infty} \sum_{n=2,4,6}^{\infty} \frac{m\pi}{2L} a_{mn} \cos\frac{m\pi x}{2L} \sin\frac{2n\pi s}{p}\right)^2 ds\,dx +$$

$$\frac{Gt}{2}\int_0^L \oint \left(\sum_{m=1,3,5}^{\infty} \sum_{n=2,4,6}^{\infty} \frac{2n\pi}{p} a_{mn} \sin\frac{m\pi x}{2L} \cos\frac{2n\pi s}{p} - \right.$$

$$\left. \rho \sum_{m=1,3,5}^{\infty} \frac{m\pi}{2L} b_m \sin\frac{m\pi x}{2L}\right)^2 ds\,dx -$$

$$\frac{I_p\omega^2}{2}\int_0^L \left(\sum_{m=1,3,5}^{\infty} b_m \cos\frac{m\pi x}{2L} + C\right)^2 dx -$$

$$\frac{\mu t_a\omega^2}{2}\int_0^L \oint \left(\sum_{m=1,3,5}^{\infty} \sum_{n=2,4,6}^{\infty} a_{mn} \sin\frac{m\pi x}{2L} \sin\frac{2n\pi s}{p}\right)^2 ds\,dx \tag{17}$$

In order to make equation (17) stationary and at the same time satisfy the constraining relationship

$$\varphi = C + \sum_{m=1,3,5}^{\infty} b_m = 0 \tag{18}$$

it is sufficient to set

$$\delta(U - T + \lambda\varphi) = 0 \tag{19}$$

where the variation is with respect to the a's, b's, and C considered as independent variables and where λ is a Lagrangian multiplier. The variational process yields the following equations:

$$\frac{\partial(U - T)}{\partial a_{ij}} = Et'\left(\frac{i\pi}{2L}\right)^2 \frac{Lp}{4} a_{ij} + Gt\left(\frac{2j\pi}{p}\right)^2 \frac{Lp}{4} a_{ij} - Gt \frac{ij\pi^2}{4} K_j b_i - \omega^2 \mu t_a \frac{Lp}{4} a_{ij} = 0 \qquad \begin{pmatrix} i = 1, 3, 5, \ldots \\ j = 2, 4, 6, \ldots \end{pmatrix} \tag{20}$$

$$\frac{\partial(U - T)}{\partial b_i} + \lambda \frac{\partial \varphi}{\partial b_i} = - \sum_{n=2,4,6}^{\infty} Gt \frac{in\pi^2}{4} K_n a_{in} + Gt\left(\frac{i\pi}{2L}\right)^2 \frac{Lp}{4} \sum_{n=2,4,6}^{\infty} K_n{}^2 b_i + Gt\left(\frac{i\pi}{2L}\right)^2 \frac{Lp}{2} K_0{}^2 b_i - \omega^2 I_p \frac{L}{2} b_i - (-1)^{\frac{i-1}{2}} \frac{2L}{i\pi} \omega^2 I_p C + \lambda = 0 \qquad (i = 1, 3, 5, \ldots) \tag{21}$$

$$\frac{\partial(U - T)}{\partial C} + \lambda \frac{\partial \varphi}{\partial C} = -\omega^2 I_p \sum_{m=1,3,5}^{\infty} \frac{2L}{m\pi}(-1)^{\frac{m-1}{2}} b_m - \omega^2 I_p LC + \lambda = 0 \tag{22}$$

where

$$K_n = \frac{2}{p} \oint \rho \cos \frac{2n\pi s}{p} \, ds \tag{23}$$

and

$$K_0 = \frac{1}{p} \oint \rho \, ds = \frac{2A_0}{p} \tag{24}$$

in which A_0 is the area enclosed by the median line of the wall thickness. In obtaining equation (21) the relation

$$\frac{1}{p} \oint \rho^2 ds = K_0^{\,2} + \frac{1}{2} \sum_{n=2,4,6}^{\infty} K_n^{\,2} \tag{25}$$

was utilized. This results from an application of Parseval's theorem, K_0 and K_n being coefficients of a Fourier series expansion of ρ.

With the use of the nondimensional parameters

$$k_T^{\,2} = \frac{I_p L^2}{GJ} \omega^2 \tag{26}$$

$$k_{LI}^{\,2} = \frac{Jp^2\mu}{I_p L^2} \tag{27}$$

$$K^2 = \frac{Ep^2}{GL^2} \tag{28}$$

and

$$B_i^{\,2} = K^2 \frac{t'}{t} i^2 - k_{LI}^{\,2} \frac{t_a}{t}\left(\frac{2}{\pi}\right)^2 k_T^{\,2} \tag{29}$$

equations (20), (21), and (22) may be reduced to

$$\left(B_i^{\,2} + 16_j^{\,2}\right)a_{ij} - 4\,\frac{p}{L}\,K_j ij b_i = 0 \qquad \begin{pmatrix} i = 1, 3, 5, \ldots \\ j = 2, 4, 6, \ldots \end{pmatrix} \tag{30}$$

$$-\frac{Lt}{4J}\,\pi^2 \sum_{n=2,4,6}^{\infty} inK_n a_{in} + \left[\frac{(i\pi)^2}{8}\left(1 + \frac{pt}{2J}\sum_{n=2,4,6}^{\infty} K_n^{\,2}\right) - \frac{1}{2}\,k_T^{\,2}\right]b_i -$$

$$\frac{2}{i\pi}\,(-1)^{\frac{i-1}{2}}\,k_T^{\,2}C + \frac{L\lambda}{GJ} = 0 \qquad (i = 1, 3, 5, \ldots) \tag{31}$$

$$k_T^{\,2}\sum_{m=1,3,5}^{\infty}\frac{2}{m\pi}\,(-1)^{\frac{m-1}{2}}\,b_m + k_T^{\,2}C - \frac{L\lambda}{GJ} = 0 \tag{32}$$

where J is the elementary torsional stiffness constant defined as

$$J = \frac{4A_o^{\,2}t}{p} \tag{33}$$

From equation (30),

$$a_{ij} = \frac{4\,\frac{p}{L}\,K_j ij}{B_i^{\,2} + 16j^2}\,b_i \qquad \begin{pmatrix} i = 1, 3, 5, \ldots \\ j = 2, 4, 6, \ldots \end{pmatrix} \tag{34}$$

Substituting this expression for a_{ij} into equation (31) and solving for b_i yields

$$b_i = \frac{\frac{2}{i\pi}\,(-1)^{\frac{i-1}{2}}\,k_T^{\,2}C - \frac{L\lambda}{GJ}}{N_i} \qquad (i = 1, 3, 5, \ldots) \tag{35}$$

where

$$N_i = \frac{(i\pi)^2}{8}\left(1 + \frac{ptB_i^{\,2}}{2J}\sum_{n=2,4,6}^{\infty}\frac{K_n^{\,2}}{B_i^{\,2} + 16n^2}\right) - \frac{1}{2}\,k_T^{\,2} \tag{36}$$

The substitution of equation (35) into equation (32) and into the constraining relationship (18) yields the following two homogeneous equations in C and λ:

$$k_T^2\left[1 + k_T^2 \sum_{m=1,3,5}^{\infty} \left(\frac{2}{m\pi}\right)^2 \frac{1}{N_m}\right]C - \left[1 + k_T^2 \sum_{m=1,3,5}^{\infty} \frac{2}{m\pi} \frac{(-1)^{\frac{m-1}{2}}}{N_m}\right]\frac{\lambda L}{GJ} = 0 \quad (37)$$

$$\left[1 + k_T^2 \sum_{m=1,3,5}^{\infty} \frac{2}{m\pi} \frac{(-1)^{\frac{m-1}{2}}}{N_m}\right]C - \left(\sum_{m=1,3,5}^{\infty} \frac{1}{N_m}\right)\frac{L\lambda}{GJ} = 0 \quad (38)$$

The condition for a nontrivial solution for C and λ gives the frequency equation

$$\begin{vmatrix} k_T^2\left[1 + k_T^2 \sum_{m=1,3,5}^{\infty} \left(\frac{2}{m\pi}\right)^2 \frac{1}{N_m}\right] & 1 + k_T^2 \sum_{m=1,2,3}^{\infty} \frac{2}{m\pi} \frac{(-1)^{\frac{m-1}{2}}}{N_m} \\ 1 + k_T^2 \sum_{m=1,3,5}^{\infty} \frac{2}{m\pi} \frac{(-1)^{\frac{m-1}{2}}}{N_m} & \sum_{m=1,3,5}^{\infty} \frac{1}{N_m} \end{vmatrix} = 0 \quad (39)$$

which the frequency parameter k_T must satisfy. Each term of each of the infinite series in the frequency equation contains k_T; therefore, the roots of equation (39) are most readily obtained by trial.

Once k_T has been determined, a particular mode shape can be found by letting $C = 1$ and solving either equation (37) or (38) for λ, and then finally evaluating b_i and a_{ij} successively from equations (35) and (34).

Free-Free Beams

Symmetrical modes.- For a free-free beam of length $2L$ with the origin at the midspan (see fig. 1), the form of the Fourier series assumed for $\theta(x)$ and $u(x,s)$ when the beam is undergoing a symmetrical mode of torsional vibration may be the same as that assumed for the

cantilever beam of length L. (See eqs. (14) and (15).) However, if consideration is given to the right half of the beam, the only geometrical boundary condition enforced is $u(0,s) = 0$; therefore, the constraining condition (18) does not apply and the appropriate frequency equation is obtained from equation (37) by letting $\lambda = 0$. The frequency equation is

$$k_T^2\left[1 + k_T^2 \sum_{m=1,3,5}^{\infty} \left(\frac{2}{m\pi}\right)^2 \frac{1}{N_m}\right] = 0 \tag{40}$$

After a particular root k_T has been found from equation (40), the corresponding mode shape may be obtained from equations (35) and (34) (with $\lambda = 0$).

Antisymmetrical modes.- For a free-free beam of length 2L vibrating in an antisymmetrical torsional mode, specific consideration need again be given only to the right half of the beam (see fig. 1). With the origin taken at the beam midspan, the only geometrical boundary condition imposed on the right half-beam is that $\theta(0) = 0$. The longitudinal displacement $u(x,s)$ is unrestricted over the length of the beam.

The displacements $\theta(x)$ and $u(x,s)$ for the antisymmetrical modes are assumed to be given by the expressions

$$\theta(x) = Cx + \sum_{m=2,4,6}^{\infty} b_m \sin \frac{m\pi x}{2L} \tag{41}$$

and

$$u(x,s) = \sum_{m=0,2,4}^{\infty} \sum_{n=2,4,6}^{\infty} a_{mn} \cos \frac{m\pi x}{2L} \sin \frac{2n\pi s}{p} \tag{42}$$

As before, the choice of the particular trigonometric series in equations (41) and (42) was guided by considerations of orthogonality. The linear term Cx appearing in equation (41) is necessary to allow sufficient freedom for the beam to twist.

Using equations (41) and (42) in equations (6) and (7) gives

$$U - T = \frac{Et'}{2}\int_0^L \oint\left(\sum_{m=2,4,6}^{\infty}\ \sum_{n=2,4,6}^{\infty} - \frac{m\pi}{2L} a_{mn} \sin\frac{m\pi x}{2L} \sin\frac{2n\pi s}{p}\right)^2 ds\ dx +$$

$$\frac{Gt}{2}\int_0^L \oint\left(\sum_{m=0,2,4}^{\infty}\ \sum_{n=2,4,6}^{\infty} \frac{2n\pi}{p} a_{mn} \cos\frac{m\pi x}{2L} \cos\frac{2n\pi s}{p} +\right.$$

$$\left.\rho \sum_{m=2,4,6}^{\infty} \frac{m\pi}{2L} b_m \cos\frac{m\pi x}{2L} + \rho C\right)^2 ds\ dx -$$

$$\frac{I_p\omega^2}{2}\int_0^L\left(Cx + \sum_{m=2,4,6}^{\infty} b_m \sin\frac{m\pi x}{2L}\right)^2 dx -$$

$$\frac{\mu t_a\omega^2}{2}\int_0^L \oint\left(\sum_{m=0,2,4}^{\infty}\ \sum_{n=2,4,6}^{\infty} a_{mn} \cos\frac{m\pi x}{2L} \sin\frac{2n\pi s}{p}\right)^2 ds\ dx \qquad (43)$$

The variation of equation (43) with respect to the a's, b's, and C gives, after simplification,

$$\left(B_i^{\,2} + 16j^2\right)a_{ij} + 4\frac{p}{L} ijK_j b_i = 0 \qquad \begin{pmatrix} i = 2, 4, 6, \ldots \\ j = 2, 4, 6, \ldots \end{pmatrix} \qquad (44)$$

$$\frac{Lt}{4J}\pi^2 \sum_{n=2,4,6}^{\infty} inK_n a_{in} + \left[\frac{(i\pi)^2}{8}\left(1 + \frac{pt}{2J}\sum_{n=2,4,6}^{\infty} K_n^{\,2}\right) - \frac{1}{2}k_T^{\,2}\right]b_i -$$

$$\left[\frac{2}{i\pi}(-1)^{\frac{i-2}{2}} k_T^{\,2}\right]LC = 0 \qquad (i = 2, 4, 6, \ldots) \qquad (45)$$

$$\left[16j^2 - \frac{t_a}{t} k_{LI}{}^2\left(\frac{2}{\pi}\right)^2 k_T{}^2\right]a_{0j} + \left(\frac{8}{\pi}\frac{p}{L} jK_j\right)LC = 0$$

$$(j = 2, 4, 6, \ldots) \qquad (46)$$

$$\frac{Lt\pi}{J}\sum_{n=2,4,6}^{\infty} nK_n a_{0n} + \left[1 + \frac{pt}{2J}\sum_{n=2,4,6}^{\infty} K_n{}^2 - \frac{1}{3} k_T{}^2\right]LC -$$

$$\frac{2}{\pi} k_T{}^2 \sum_{m=2,4,6}^{\infty} \frac{(-1)^{\frac{m-2}{2}}}{m} b_m = 0 \qquad (47)$$

From equation (44),

$$a_{ij} = \frac{-4\frac{p}{L}K_j ij}{B_i{}^2 + 16j^2} b_i \qquad \begin{pmatrix} i = 2, 4, 6, \ldots \\ j = 2, 4, 6, \ldots \end{pmatrix} \qquad (48)$$

which, except for sign, is the same expression as that obtained for the cantilever and symmetrically vibrating free-free beams (eq. (34)). From equation (46),

$$a_{0j} = \frac{-\frac{8}{\pi}\frac{p}{L}K_j j}{16j^2 + B_0{}^2} LC \qquad \left(j = 2, 4, 6, \ldots\right) \qquad (49)$$

Substituting equation (48) into equation (45) and solving for b_i gives

$$b_i = \frac{\frac{2}{\pi}(-1)^{\frac{i-2}{2}} k_T{}^2}{iN_i} LC \qquad \left(i = 2, 4, 6, \ldots\right) \qquad (50)$$

where N_i is given by equation (36).

Substitution of equations (49) and (50) into equation (47) and simplification gives the following frequency equation for the antisymmetrical vibrating free-free beam:

$$k_T^2\left[\frac{pt_a\mu}{2I_p}\left(\frac{2p}{\pi L}\right)^2 \sum_{n=2,4,6}^{\infty} \frac{K_n^2}{B_0^2 + 16n^2} + \left(\frac{2}{\pi}\right)^2 k_T^2 \sum_{n=2,4,6}^{\infty} \frac{1}{n^2N_n} + \frac{1}{3}\right] = 1 \qquad (51)$$

After a particular value of k_T is obtained from equation (51), the corresponding mode shape may be computed by letting $C = 1$ and calculating the b's and a's successively from equations (50), (49), and (48).

Discussion of Parameters

In the frequency equation, the unknown natural frequency appears only in the frequency coefficient k_T which is defined by the commonly used relationship $\omega = k_T\sqrt{\frac{GJ}{I_pL^2}}$. The parameter K is associated with the effect of bending stresses due to torsion, and, if this effect is to be neglected, it is sufficient to set K equal to zero in the final frequency equation. The parameter k_{LI} is associated with the effects of longitudinal inertia and appears only in the expression for B_i. (See eq. (29).) If the effects of longitudinal inertia are to be neglected, k_{LI} may be set equal to zero. With $k_{LI} = 0$, B_i is independent of k_T and simplification of the trial-and-error solution for the natural frequencies results, since the infinite summation contained in N_i (eq. (36)) need be calculated only once for a particular beam. In the following section, it is shown that the influence of longitudinal inertia is often negligible.

The K_n's are the Fourier coefficients of the function $\rho(s)$, which depends only on the contour of the cross section of the beam. It is interesting to note that, for a cylinder whose cross section is either circular or a regular polygon, the K_n's become zero and the frequency equations (39), (40), and (51) can be written in a closed form identical with the frequency equations obtained from elementary theory. This is consistent with the generally known fact that beams of such cross section have no tendency to warp; hence there can be no longitudinal inertia or bending stresses due to torsion.

RESULTS FOR CYLINDRICAL BEAMS OF RECTANGULAR CROSS SECTION

In order to illustrate the effects of bending stresses due to torsion and longitudinal inertia on the natural torsional frequencies, the present analysis has been applied to hollow thin-walled cylindrical beams of rectangular cross section. The beams considered are assumed to have walls of constant thickness, with $t' = t = t_a$, and a cross-sectional aspect ratio of 3.6. The numerical calculations have been limited to the free-free symmetrical modes of vibration. A value of E/G of 2.65 (appropriate for aluminum alloys) was used. Results in the form of the variation of the frequency coefficient k_T with plan-form aspect ratio L/b are shown in figure 2 for the first four modes of vibration.

The short-dashed curves in figure 2 show results obtained from calculations based on elementary torsion theory where the restraint due to warping of the cross section is neglected. These curves serve as a basis for resolving the effects of bending stresses due to torsion. The long-dashed curves are calculated from equation (40) with $k_{LI} = 0$ and, consequently, represent the natural torsional frequencies when only the effects of bending stresses due to torsion are included. (For the rectangular tube the quantity N_1 that appears in equation (40) can be put in the closed form presented in appendix A; this closed form was used in this calculation as well as in the calculation of the exact frequencies mentioned in the next paragraph.) Comparison of these two sets of curves shows that, as might be expected, the effect of including bending stresses due to torsion is to increase the calculated natural frequency of the beam. For the higher plan-form aspect ratios, the contribution of bending stresses due to torsion is small. However, this contribution does become appreciable at the lower values of L/b for the higher modes of vibration.

The solid curves show the calculated results for the first four modes, when the combined influence of longitudinal inertia and bending stresses due to torsion is taken into account. The values of k_T shown by these curves, which are referred to as "exact," are the lowest four roots of equation (40).

Comparison of these curves with the long-dashed curves shows that, as should be expected, the effect of including longitudinal inertia is to decrease the calculated torsional frequencies. For plan-form aspect ratios L/b above 4, this effect is practically negligible. For the lower plan-form aspect ratios, however, the effect becomes appreciable; the exact curves no longer follow the corresponding long-dashed curves, but drop off in a seemingly disorderly manner.

The explanation for the apparently erratic behavior of the exact curves may be obtained by considering the kinetic energies included in the derivation of the exact frequency equation (eq. (40)). As was previously mentioned, the kinetic energy (eq. (7)) consists of two parts: one which represents the contribution of the rotational inertia and another which represents the contribution of the longitudinal inertia. If only the inertia of rotation is considered, the resulting frequency equation represents a primarily torsional vibration in which only the effects of bending stresses due to torsion are included. (The results of such a solution have already been shown by the long-dashed curves in fig. 2.) If, however, only the longitudinal inertia is considered, the frequency equation obtained represents a particular type of primarily longitudinal vibration which is hereinafter referred to as "warping" vibration. (It should be noted that this vibration is not entirely longitudinal; some twist is present, just as longitudinal motion is present in the torsional vibration.) The exact frequency equation, which includes the effects of both the rotational and the longitudinal inertia, can be looked upon as representing the coupling of the two "uncoupled" vibrations (torsional and warping).

In order to illustrate this coupling, the frequencies of the warping vibration have been obtained in appendix B and are included in figure 2 as the dot-dash lines. Consider now, for example, the fourth exact frequency. As the plan-form aspect ratio is decreased, the exact frequency follows the fourth uncoupled torsional frequency until an L/b value of about 3.5 is reached. At this point, the effect of the warping vibrations becomes predominant and the curve falls off rapidly until the third uncoupled torsional frequency is reached. (Presumably, the mode shape of the vibration during this transit between the uncoupled torsional frequencies is roughly the same as the first warping mode.) As L/b is decreased further, the curve continues to cascade in a similar manner.

Thus the behavior of the exact frequency curves is seen to be quite rational. At a given value of L/b, the exact frequency coefficients and their associated modes can be interpreted by considering the frequency coefficients of both uncoupled torsional and warping vibrations. At $L/b = 2$, for example, the first, second, and fourth exact frequencies are comparable, respectively, to the first, second, and third uncoupled torsional frequencies. The third exact frequency, which appears to be an extra torsional frequency, is associated with the first warping mode of vibration. This same phenomenon of seemingly extra frequencies due to inclusion of longitudinal inertia was noted for bending vibrations by Traill-Nash and Collar in reference 2.

The disposition of the exact curves in figure 2 with respect to the approximate curves suggests that it may not be necessary to consider the problem of coupled torsion and warping in order to obtain natural frequencies of reasonable accuracy. Certainly, for the higher values of L/b, the frequency coefficients obtained by neglecting longitudinal inertia are sufficiently accurate. For the lower values of L/b, the

transitions of the exact frequencies can be characterized by considering, in addition, the frequencies of uncoupled warping vibration.

Besides showing the quantitative influence of the secondary effects, the present analysis can be put to another practical use: to assess the accuracy of simplified torsional-vibration theories. One such theory is presented in appendix C and is based on an analysis of a four-flange box such as is shown in figure 3. In such a box beam, the flanges are assumed to carry only normal stresses while the sheets carry only shear. In order to represent the rectangular cylinder as a four-flange beam, the cross-sectional area of each flange is taken to be equal to 1/6 the section area of the walls adjacent to each corner of the box. In the analysis, the effects of longitudinal inertia are neglected.

The results of calculations for the natural torsional frequencies of rectangular tubes based on the four-flange solution are shown in table 1. For comparison, the frequency coefficients calculated from the elementary solution and from the exact solution without longitudinal inertia are included. Examination of the results shows that the four-flange solution gives frequencies that are in good agreement with those of the exact solution, even for the low-aspect-ratio box beam where the influence of bending stresses due to torsion is important.

CONCLUDING REMARKS

The results of numerical calculations show that elementary theory is adequate for the analysis of torsional vibrations of box beams with high or medium plan-form aspect ratios. For box beams with low plan-form aspect ratios, the influence of bending stresses due to torsion becomes important. The influence of longitudinal inertia, however, does not become important until extremely low (and probably impractical) plan-form aspect ratios are reached. A simplified procedure for obtaining torsional frequencies, based on an analysis of an equivalent four-flange box beam, was found to give adequate predictions of the effects of bending stresses due to torsion. The general analyses presented, as well as the numerical results for the rectangular box beams, should be useful in the assessment of the accuracy of other simplified procedures that may be developed.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., May 12, 1954.

APPENDIX A

CLOSED FORM OF N_i FOR CYLINDRICAL BEAMS OF RECTANGULAR CROSS SECTION

The expression for N_i can be written in closed form for a cylindrical beam of rectangular cross section and constant thickness. For such a beam with the dimensions shown in figure 2, the parameters become

$$\left.\begin{aligned} p &= 4(a + b) \\ J &= \frac{16a^2b^2t}{a + b} \\ K_n &= \frac{4}{\pi}\frac{a - b}{n}\sin\frac{2n\pi b}{p} \qquad (n \text{ even}) \\ K_n &= 0 \qquad (n \text{ odd}) \end{aligned}\right\} \tag{A1}$$

With equation (A1), equation (36) can be written as

$$N_i = \frac{(i\pi)^2}{8}\left[1 + \frac{2\left(a^2 - b^2\right)^2}{\pi^2a^2b^2} B_i^2 \sum_{n=2,4,6}^{\infty} \frac{\sin^2\frac{n\pi}{2\left(\frac{a}{b} + 1\right)}}{n^2\left(B_i^2 + 16n^2\right)}\right] - \frac{1}{2}k_T^2 \tag{A2}$$

or

$$N_i = \frac{(i\pi)^2}{8}\left[1 + 2\left(\frac{a^2 - b^2}{\pi ab}\right)^2\left(\sum_{n=2,4,6}^{\infty} \frac{\sin^2\frac{n\pi}{2\left(\frac{a}{b} + 1\right)}}{n^2} - \sum_{n=2,4,6}^{\infty} \frac{\sin^2\frac{n\pi}{2\left(\frac{a}{b} + 1\right)}}{\frac{B_i^2}{16} + n^2}\right)\right] - \frac{1}{2}k_T^2 \tag{A3}$$

The infinite summations in equation (A3) can be written in closed form and the resulting closed expression for N_i is

$$N_i = \frac{(i\pi)^2}{8}\left[1 + \left(\frac{a^2 - b^2}{ab}\right)^2\left(\frac{ab}{4(a + b)^2} + \frac{1}{\pi B_i}\,\frac{\cosh \frac{\pi B_i}{8}\,\frac{a - b}{a + b} - \cosh \frac{\pi B_i}{8}}{\sinh \frac{\pi B_i}{8}}\right)\right] - \frac{1}{2}\,{k_T}^2 \tag{A4}$$

APPENDIX B

SOLUTION FOR THE WARPING VIBRATION OF CYLINDRICAL BEAMS

The solution for the warping vibration of thin-walled cylindrical beams can be obtained from the variational equation, as was mentioned previously, by setting the first term in the expression for kinetic energy (see eq. (7)) equal to zero.

In order to obtain the antisymmetrical warping modes that couple with the free-free symmetrical torsional modes of cylinders with uniform wall thickness, it is sufficient to set I_p equal to 0 in equation (17); then

$$U - T = \frac{Et'}{2}\int_0^L \oint \left(\sum_{m=1,3,5}^{\infty} \sum_{n=2,4,6}^{\infty} \frac{m\pi}{2L} a_{mn} \cos\frac{m\pi x}{2L} \sin\frac{2n\pi s}{p}\right)^2 ds\, dx +$$

$$\frac{Gt}{2}\int_0^L \oint \left(\sum_{m=1,3,5}^{\infty} \sum_{n=2,4,6}^{\infty} \frac{2n\pi}{p} a_{mn} \sin\frac{m\pi x}{2L} \cos\frac{2n\pi s}{p} - \right.$$

$$\left. \rho \sum_{m=1,3,5}^{\infty} \frac{m\pi}{2L} b_m \sin\frac{m\pi x}{2L}\right)^2 ds\, dx -$$

$$\frac{\mu t_a \omega^2}{2}\int_0^L \oint \left(\sum_{m=1,3,5}^{\infty} \sum_{n=2,4,6}^{\infty} a_{mn} \sin\frac{m\pi x}{2L} \sin\frac{2n\pi s}{p}\right)^2 ds\, dx \qquad \text{(B1)}$$

The substitution of equation (B1) into equation (5) yields the following equations:

$$\frac{\partial(U - T)}{\partial a_{ij}} = \left(B_i^2 + 16j^2\right)a_{ij} - 4\frac{p}{L} K_j ij b_i = 0 \quad \begin{pmatrix} i = 1, 3, 5, \ldots \\ j = 2, 4, 6, \ldots \end{pmatrix} \qquad \text{(B2)}$$

$$\frac{\partial(U - T)}{\partial b_i} = - \sum_{n=2,4,6}^{\infty} \frac{Lt}{4J} \pi^2 i n K_n a_{in} + \frac{(i\pi)^2}{8}\left(1 + \frac{pt}{2J} \sum_{n=2,4,6}^{\infty} K_n^2\right) b_i = 0$$

$$(i = 1, 3, 5, \ldots) \qquad \text{(B3)}$$

If equation (B2) is solved for a_{ij} and the resulting expression is substituted into equation (B3), the following equation is obtained:

$$\frac{(i\pi)^2}{8}\left(1 + \frac{pt}{2J} \sum_{n=2,4,6}^{\infty} K_n^2 - \frac{8pt}{J} \sum_{n=2,4,6}^{\infty} \frac{n^2 K_n^2}{B_i^2 + 16n^2}\right) b_i = 0 \qquad \text{(B4)}$$

In order that nonzero values of b_i may exist, the frequency equation

$$1 + \frac{pt}{2J} B_i^2 \sum_{n=2,4,6}^{\infty} \frac{K_n^2}{B_i^2 + 16n^2} = 0 \qquad \text{(B5)}$$

must be satisfied. The frequency coefficient k_T is contained in the parameter B_i. Once a particular root k_T has been found from equation (B5), the corresponding mode shape can be obtained from equation (B2) by setting b_i equal to the arbitrary value of 1.

APPENDIX C

SOLUTION FOR A FOUR-FLANGE BOX BEAM

A simplified torsion theory, used to obtain the torsional modes and frequencies of structures where the influence of bending stresses due to torsion is to be included, can be based on an analysis of a four-flange box beam such as that shown in figure 3. (The torsional vibrations of such a structure were considered in 1951 by E. H. Mansfield of the Royal Aircraft Establishment, who obtained results for a cantilever beam and also for an antisymmetrically vibrating free-free beam; in this appendix results will be shown for the free-free symmetrical modes.) In an analysis of such a structure the following assumptions are made:

(a) The flanges are assumed to carry only normal stresses.

(b) The sheets connecting the flanges are assumed to carry only shear.

(c) The cross sections remain undistorted.

(d) Longitudinal inertia is assumed to be negligible.

The actual flange of the box beam should be replaced by an effective flange which incorporates all of the normal-stress carrying capacity of the beam.

In agreement with these assumptions, the distortion of the vibrating beam is defined by a rotation $\theta(x)$ of the cross section and a longitudinal displacement $u_F(x)$ of each flange.

The longitudinal and shear strains are given by equations (1) and (2) with

$$\frac{\partial u}{\partial s} = \frac{u_F}{b} \tag{C1}$$

and

$$\rho = a \tag{C2}$$

for the cover sheets and

$$\frac{\partial u}{\partial s} = -\frac{u_F}{a} \tag{C3}$$

and

$$\rho = b \tag{C4}$$

for the webs.

The solution for the four-flange box beam could be obtained by means of the variational principle; however, the direct use of the integrodifferential equations is more expedient.

Substituting equations (C1) to (C4) into equation (9) yields

$$-4Gt(b - a)u_F' + 4Gtab(a + b)\theta'' + \omega^2 I_p\theta = 0 \tag{C5}$$

where primes denote differentiation with respect to x. Equation (C5) can be recognized as the equilibrium equation for a cross-sectional element of the beam.

Because of the assumption that the sheet in a four-flange box carries only shear, equation (8) has meaning only at the flanges of the box. Since the flanges are assumed to be concentrated at a point, the thickness t' in equation (8) must be defined as

$$t' = \delta(s - b) \tag{C6}$$

where δ is the Dirac delta function defined by

$$\left.\begin{aligned} \delta(s - b) &= 0 \qquad (s \neq b) \\ \delta(s - b) &= \infty \qquad (s = b) \end{aligned}\right\} \tag{C7}$$

in such a way that

$$\int_{b-\Delta}^{b+\Delta} \delta(s - b)ds = A_F \tag{C8}$$

Therefore, if equation (8) is integrated over the corners, that is, from $s = b - \Delta$ to $s = b + \Delta$, where the infinitesimal quantity Δ approaches zero, the following differential equation is obtained:

$$EA_F u_F'' - Gt\left(\frac{1}{a} + \frac{1}{b}\right)u_F + Gt(b - a)\theta' = 0 \tag{C9}$$

Equation (C9) is the equilibrium equation for an element of a flange.

In order to obtain the equilibrium equations (C5) and (C9), use was made of the previously derived integrodifferential equations (8) and (9) to show the applicability of these general equations to a particular problem. However, equations (C5) and (C9) could have been written directly from consideration of the equilibrium of the various elements.

Either θ or u_F may be eliminated from equations (C5) and (C9) to give the following equation in θ or u_F alone:

$$\frac{d^4\theta}{d\xi^4} + \left[k_T^2 \frac{4ab}{(a+b)^2} - \frac{4GtL^2}{EA_F(a+b)}\right]\frac{d^2\theta}{d\xi^2} - k_T^2 \frac{4GtL^2}{EA_F(a+b)}\theta = 0 \qquad \text{(C10)}$$

or

$$\frac{d^4u_F}{d\xi^4} + \left[k_T^2 \frac{4ab}{(a+b)^2} - \frac{4GtL^2}{EA_F(a+b)}\right]\frac{d^2u_F}{d\xi^2} - k_T^2 \frac{4GtL^2}{EA_F(a+b)}u_F = 0 \qquad \text{(C11)}$$

where

$$\xi = \frac{x}{L}$$

Solutions of equations (C10) and (C11) are of the form $e^{\alpha\xi}$ where α is a root of the equation

$$\alpha^4 + \left[k_T^2 \frac{4ab}{(a+b)^2} - \frac{4GtL^2}{EA_F(a+b)}\right]\alpha^2 - k_T^2 \frac{4GtL^2}{EA_F(a+b)} = 0 \qquad \text{(C12)}$$

The general solutions for θ and u_F are given by

$$\theta = C_1 \sinh \alpha_1\xi + C_2 \cosh \alpha_1\xi + C_3 \sin \alpha_2\xi + C_4 \cos \alpha_2\xi \qquad \text{(C13)}$$

and

$$u_F = D_1 \sinh \alpha_1\xi + D_2 \cosh \alpha_1\xi + D_3 \sin \alpha_2\xi + D_4 \cos \alpha_2\xi \qquad \text{(C14)}$$

where α_1 and α_2 are the real and imaginary roots, respectively, of equation (C12). The constants D_1 to D_4 are related to the constants C_1 to C_4 through equation (C5) or (C9) as follows:

$$\left.\begin{aligned}
\frac{L^2}{4}\frac{b^2 - a^2}{a^2b^2}\alpha_1\frac{D_1}{L} - \left(B\alpha_1^{\ 2} + k_T^{\ 2}\right)C_2 &= 0 \\
\frac{L^2}{4}\frac{b^2 - a^2}{a^2b^2}\alpha_1\frac{D_2}{L} - \left(B\alpha_1^{\ 2} + k_T^{\ 2}\right)C_1 &= 0 \\
\frac{L^2}{4}\frac{b^2 - a^2}{a^2b^2}\alpha_2\frac{D_3}{L} + \left(B\alpha_2^{\ 2} - k_T^{\ 2}\right)C_4 &= 0 \\
\frac{L^2}{4}\frac{b^2 - a^2}{a^2b^2}\alpha_2\frac{D_4}{L} - \left(B\alpha_2^{\ 2} - k_T^{\ 2}\right)C_3 &= 0
\end{aligned}\right\} \tag{C15}$$

where

$$B = \frac{(a + b)^2}{4ab} \tag{C16}$$

In addition to equations (C15), four other relations between the constants are needed and these relations are obtained from the boundary conditions imposed on the beam.

For a free-free beam undergoing symmetrical vibrations, the boundary conditions are at the midspan

$$\left.\begin{aligned}
\frac{d\theta}{dx} &= 0 \\
u_F &= 0
\end{aligned}\right\} \tag{C17}$$

and at the free end

$$\frac{du_F}{dx} = 0 \tag{C18a}$$

$$(a - b)u_F + ab(a + b)\frac{d\theta}{dx} = 0 \tag{C18b}$$

where equation (C18b) is the condition of zero torque.

The conditions (C17) and (C18) can be expressed in terms of either set of constants, by the use of equations (C15). The condition for a nontrivial solution gives the frequency equation

$$\begin{vmatrix} 0 & \left(B\alpha_2^2 - k_T^2\right)\alpha_1^2 & 0 & \left(B\alpha_1^2 + k_T^2\right)\alpha_2^2 \\ 0 & 1 & 0 & 1 \\ \alpha_1 \cosh \alpha_1 & \alpha_1 \sinh \alpha_1 & \alpha_2 \cos \alpha_2 & -\alpha_2 \sin \alpha_2 \\ \left(k_T^2 - B\alpha_2^2\right)\sinh \alpha_1 & \left(k_T^2 - B\alpha_2^2\right)\cosh \alpha_1 & \left(k_T^2 + B\alpha_1^2\right)\sin \alpha_2 & \left(k_T^2 + B\alpha_1^2\right)\cos \alpha_2 \end{vmatrix} = 0 \qquad \text{(C19)}$$

which reduces to

$$\left(k_T^2 + B\alpha_1^2\right)\alpha_1 \cosh \alpha_1 \sin \alpha_2 - \left(k_T^2 - B\alpha_2^2\right)\alpha_2 \sinh \alpha_1 \cos \alpha_2 = 0 \qquad \text{(C20)}$$

The frequency equation has been obtained for the special case of a rectangular box beam with web and cover sheets of equal thickness in order to compare the numerical results from the four-flange solution with the results from the exact solution. However, the solution for the four-flange box beam with unequal web and cover sheets or with other boundary conditions may be obtained by means of the same analysis procedure.

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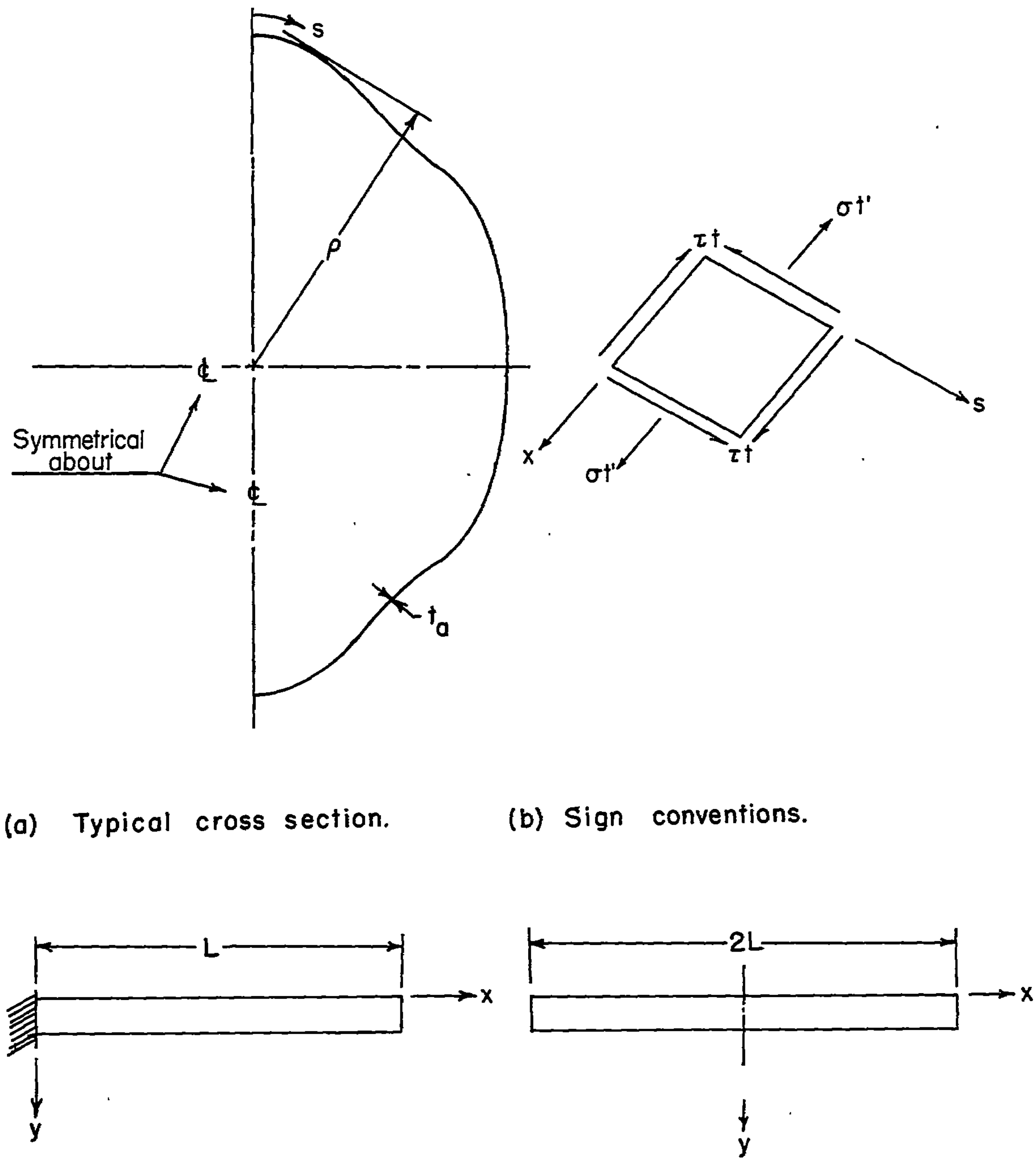
TABLE 1

COMPARISON OF RESULTS OBTAINED FROM FOUR-FLANGE SOLUTION WITH THOSE OBTAINED FROM ELEMENTARY AND EXACT SOLUTIONS

$$\left[\frac{b}{a} = 3.6\right]$$

L/b	Elementary solution	Exact solution without longitudinal inertia	Four-flange solution
First mode k_T			
2	3.14	3.30	3.31
6	3.14	3.18	3.18
10	3.14	3.16	3.16
14	3.14	3.15	3.15
Second mode k_T			
2	6.28	6.98	7.11
6	6.28	6.53	6.54
10	6.28	6.39	6.40
14	6.28	6.35	6.35
Third mode k_T			
2	9.42	10.80	11.01
6	9.42	10.02	10.13
10	9.42	9.78	9.77
14	9.42	9.65	9.63



(a) Typical cross section.

(b) Sign conventions.

(c) Cantilever beam.

(d) Free-free beam.

Figure 1.— Coordinate systems and sign conventions.

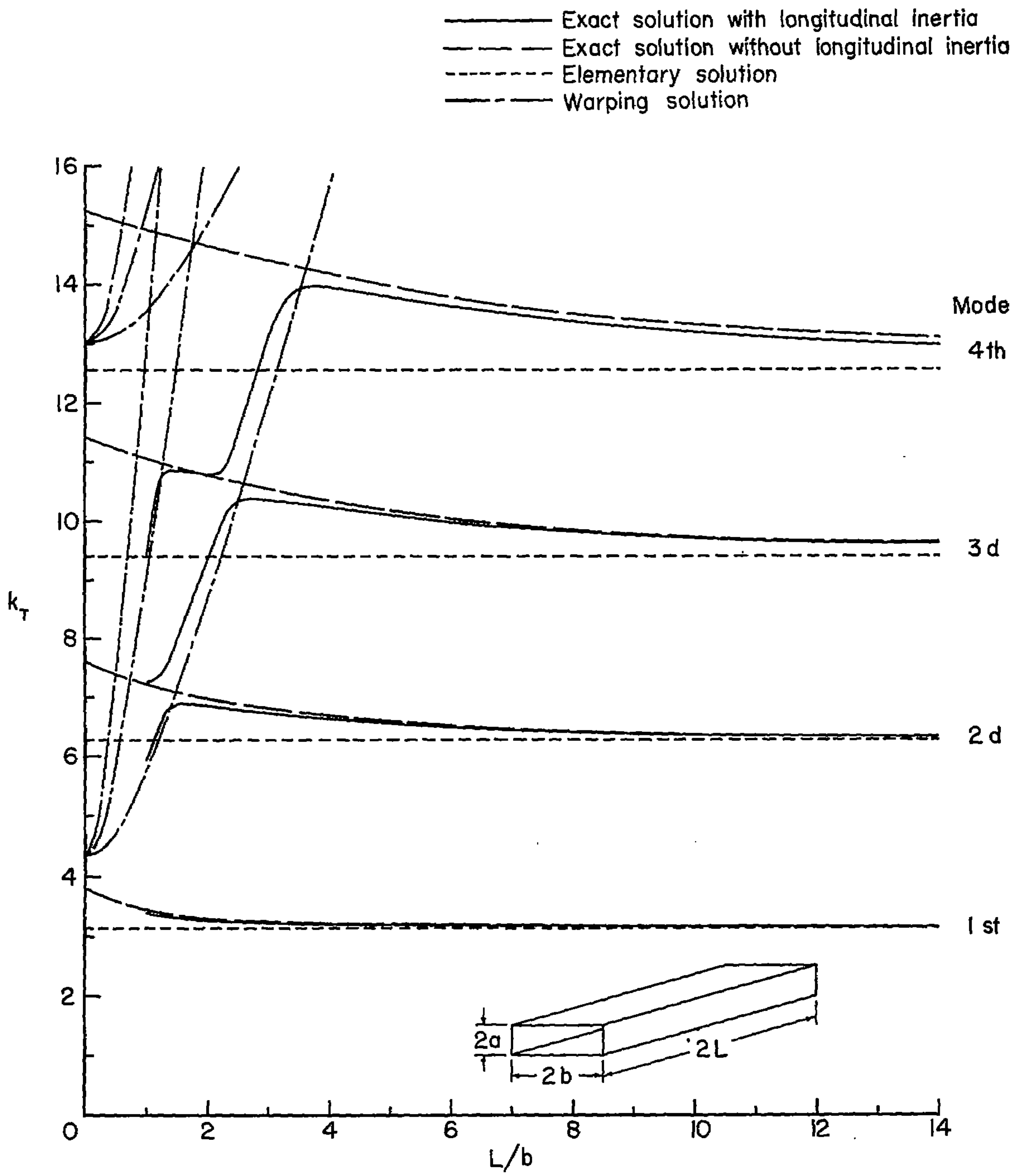


Figure 2.- Influence of plan-form aspect ratio on frequency coefficient for free-free symmetrical modes, $\frac{b}{a}=3.6$.

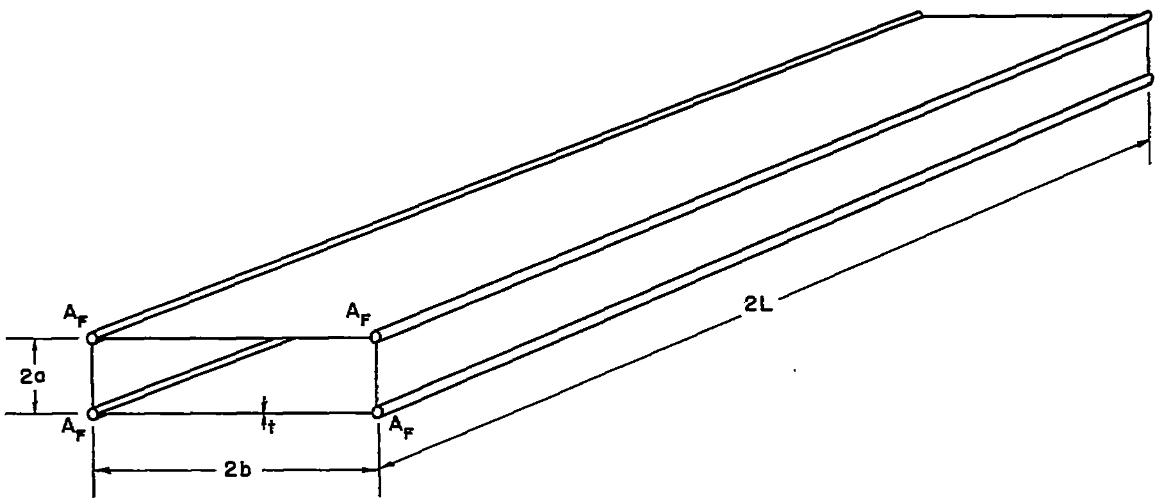


Figure 3.- Four-flange box beam.

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